Editorial

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In the last ten years, an intense activity has been modifying and improving our understanding of contemporary Statistical Mechanics and Thermodynamics. After the success of NEXT2001, held in Sardinia in September 2001, where a large variety of systems whose description appears to require the extension of the Boltzmann-Gibbs formalism were discussed [1], this second conference on "News and expectations in thermostatistics" NEXT2003 has brought together and synthesized the latest work in a field where theory has reached a critical turning point and several still unclear points have acquired sound fundamental justifications. In addition to a critical assessment of the results achieved, this conference has helped cross-fertilization of interdisciplinary and new applications and an unbiased discussion.

During the Conference about one third of the presentations dealt with fundamentals and theoretical methods of modern statistical mechanics and thermodynamics. All the other presentations focused on applications of statistical mechanics to specific fields: Hamiltonian systems, chaos, fluctuations, stochastic systems, time series, models of complex and quantum systems, gravitation and high energy physics.

As these proceedings together with those of the previous edition of the conference [1] give a good overview of the large activity and variety of results in the field, we devote the editorial to a historical outline of the development of some mathematical functions that play an important role in the current theoretical foundations of the field.

In 1779, three years before his death, Euler studied the properties of a series first introduced by Lambert in 1758 in order to express roots of the trinomial algebraic equation $x = q + x^m$. In his paper *De serie Lambertina plurimisque eius insignibus proprietatibus* [2] Euler considered the *aequatio trinomialis* in the symmetrical form $x^\alpha - x^\beta = (\alpha - \beta)v\, x^{\alpha+\beta}$. Here he introduced the function

$$v = \frac{x^{-\beta} - x^{-\alpha}}{\alpha - \beta} \tag{1}$$

and its derivative. In particular he illustrated the special case $\beta = 0$ and the limiting case $\beta = \alpha$.

When discussing $\beta = 0$, the function defined in Eq. (1) becomes:

$$v = \frac{1 - x^{-\alpha}}{\alpha} \ , \tag{2}$$

and Euler found its inverse:

$$x = (1 - \alpha\, v)^{-\frac{1}{\alpha}} \ ; \tag{3}$$

while, when studying the case $\beta = \alpha$, he put $\alpha = \beta + \omega$ with $\omega \to 0$ (ω *infinite parvo*) to avoid an indeterminate form of the equation and explicitly uses (page 353 of Ref. [2]) the limit: *Constat autem evanescente* ω *esse*

$$\frac{x^\omega - 1}{\omega} = lx \ , \tag{4}$$

where symbol lx indicates the natural logarithm $\ln(x)$.

Euler used (1),(2) and (3), and the limit (4) as mathematical tools to approach a specific mathematical problem: after 130 years these same tools reappeared in mathematical statistics and, 60 years later, in information theory and statistical physics.

Indeed in 1908 W. S. Gosset, under the pen name *Student*, published [3] the distribution function:

$$y = A_n\, (1 + z^2)^{-\frac{1}{2}n} \ , \tag{5}$$

where the free parameter n is an integer. Guided by the distribution (5) and by the subsequent work in mathematical statistics, in 1968 V. M. Vasyliunas [4] introduced the phenomenological distribution:

$$y = C_\kappa \left(1 + \frac{v^2}{\kappa\, \theta^2}\right)^{-(\kappa+1)} \tag{6}$$

to reproduce cosmic ray energy distributions: Eq. (6) goes smoothly to the Maxwellian distribution when the continuous parameter κ of Vasyliunas approaches infinity. Nowadays distribution (6) is widely used in plasma physics where it is known as κ-distribution.

Before discussing the successive applications of Euler's Eqs. (1)-(4) in statistical mechanics and information theory, we should recall a few points concerning entropy. A generalized trace-form entropy can be written as:

$$\mathcal{S}(p) = \sum_i p_i \, f(1/p_i) \ , \tag{7}$$

where $f(x)$ is an arbitrary function whose properties have been studied by Csiszar [6]. This function can be viewed as a generalization of the logarithm since, when $f(x) = \ln x$, the above generalized entropy reduces to the Boltzmann-Gibbs-Shannon entropy. After the publication in 1948 of Shannon's paper [5], other entropies have been proposed within information theory: many of them are of the form (7) for specific choices of $f(x)$.

In information theory and mathematical statistics parameter interpretation is less important and Boltzmann-Gibbs-Shannon entropy has been generalized in about thirty different ways adding one or two parameters. Distribution functions are even more diversified: a handbook of generalized special functions for statistical and physical sciences by Mathai [13] lists more than 130 different statistical distributions.

In 1967 Harvda and Charvat proposed a one-parameter generalization of Shannon information entropy of the form (7) with a choice of $f(x)$ which is basically Eq. (2).

In 1975, in the context of information theory, Mittal [9] and Sharma and Taneja [10] proposed a further generalization of the entropy of Harvda and Charvat, using as $f(x)$ the two-paramter dependence (1). This entropy has not been as successful as the one of Harvda-Charvat.

In 1988 Tsallis, motivated by multi fractal scaling, first postulated the physical relevance of a one-parameter generalization of the entropy [8], whose form is equivalent to that of Harvda and Charvat. This seminal paper *opens the door to the generalization of standard thermodynamics* and of the Boltzmann-Gibbs statistical mechanics.

In Tsallis' theory, which is often referred to as non-extensive statistical mechanics, generalizations of the logarithm and of the exponential have an important role:

$$\ln_q(x) = \frac{x^{1-q}-1}{1-q} \ , \qquad \exp_q(x) = [1+(1-q)x]_+^{1/(1-q)} \ , \tag{8}$$

where $[x]_+ \equiv \max(x, 0)$. These two functions, which coincide with Eqs. (2) and (3), have appeared for the first time in the same coherent physical context linked by the maximum entropy principle.

Tsallis' entropy:

$$\mathcal{S}_q(p) = \sum_i p_i \ \ln_q(1/p_i) \ , \tag{9}$$

and the related distribution function expressed in terms of $\exp_q(x)$ represent the foundation of the non-extensive statistical mechanics: the reader is referred to the large number of papers that deal with this theory and its applications, both in this volume and in the previous one [1], for a more complete discussion.

In 1997 Abe generalized statistical mechanics in a new direction [11] within the framework of quantum groups. Soon afterwards Borges and Roditi [12] found a unified formulation for Tsallis' and Abe's entropy. They proposed a two-parameter entropy that happened to be that of Mittal-Sharma-Taneja. In fact Borges and Roditi have constructed with this entropy a generalized statistical mechanics that yields a two-parameter distribution which decays as a power law. Only for special choices of the parameters, such as the choice that yields Tsallis' statistical mechanics, this distribution can be obtained explicitly without resorting to numerical evaluation.

The distribution function corresponding to Mittal-Sharma-Taneja entropy decays as a power law: one of the two parameters is immediately related to this asymptotic behavior, while the second one is of more difficult interpretation. This fact might explain why this two-parameter entropy has not been used by the physics community. A simple expression for the entropy and the distribution function and a minimal number of parameters are important features of a physical theory.

Very recently [14] the requirement that the generalized logarithm verifies $f(1/x) = -f(x)$ has produced a new family of trace-form entropies where in Eq. (7) $f(x) = g\left((x^\kappa - x^{-\kappa})/(2\kappa)\right)$ with $g(x)$ an arbitrary, odd, and increasing function. For the special case $g(x) = x$ one obtains the following expressions for the generalized logarithm and its inverse function, *i.e.*, the generalized exponential:

$$\ln_{\{\kappa\}}(x) = \frac{x^\kappa - x^{-\kappa}}{2\kappa} \, , \qquad \exp_{\{\kappa\}}(x) = \left(\sqrt{1+\kappa^2x^2} + \kappa x\right)^{1/\kappa} \quad . \tag{10}$$

Using the one-parameter generalized entropy

$$\mathcal{S}_\kappa(p) = \sum_i p_i \ \ln_{\{\kappa\}}(1/p_i) = -\sum_i p_i \ \ln_{\{\kappa\}}(p_i) \ , \tag{11}$$

the maximum entropy principle yields the one-parameter distribution function:

$$p_i = \alpha \exp_{\{\kappa\}} \left(-\frac{E_i - \mu}{\lambda T} \right) \quad , \tag{12}$$

which decays asymptotically as a power law and reproduces the Maxwell-Boltzmann distribution in the limit $\kappa \to 0$. The structure of the ensuing generalized statistical mechanics has striking similarity with that of special relativity suggesting that it might be relevant for systems where information propagates with finite speed, *e.g.*, relativistic particle systems [14].

As useful mathematical tools, one could find many two-parameter generalizations of the logarithm that reproduce as special cases the $q-$logarithm or the $\kappa-$logarithm. A simple two-parameter logarithm, which has the important property that its inverse exists in terms of elementary functions, is the scaled $\kappa-$logarithm

$$\ln_{\{\kappa,\varsigma\}}(x) \equiv \frac{2}{\varsigma^{\kappa} + \varsigma^{-\kappa}} \left[\, \ln_{\{\kappa\}}(\varsigma x) - \ln_{\{\kappa\}}(\varsigma) \, \right] \ , \tag{13}$$

where ς is the scaling parameter. Note that this two-parameter generalization of the logarithm, $\ln(x) = \ln_{\{0,\varsigma\}}(x)$, contains as special cases both the $q-$logarithm, $\ln_q(x) = \ln_{\{q-1,0\}}(x)$ and the $\kappa-$logarithm, $\ln_{\{\kappa\}}(x) = \ln_{\{\kappa,1\}}(x)$.

A second example of two-parameter generalization of the logarithm is given by Euler's function (1), which includes both the q-logarithm and the κ-logarithm. This latter special case, the κ-logarithm, has not been discussed either by Euler, or by Mittal [9], or by Sharma-Taneja [10], or by Borges-Roditi [12], when using (1). Nor has the κ-exponential, in spite of its simplicity, attracted the attention of researchers or been included in the large compilation of statistical distributions by Mathai [13].

We observe that, starting from Euler's work, the fundamental limit (4), on which the definition of natural logarithm and the constant e itself are based, has been used by generations of scientists and students. After about two centuries, the founders of generalized information theory and non-extensive statistical mechanics resorted to this same formula to generalize the entropy of Boltzmann-Gibbs-Shannon paying an implicit tribute to Euler, one of the most prolific mathematician in history.

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